

Kirby calculus?

Handlebodies on smooth manifolds were Smale's way of looking at the Morse theory; he used them beautifully in his proof of the high dimensional h-cobordism theorem. Nowadays some authors call 4-dimensional handlebodies¹ "Kirby diagrams" and handle slides "Kirby calculus"

In his 1974 seminal paper "A calculus of framed links in S^3 " (which was published in 1978 in "Invent. Math") Kirby showed how two handlebodies describing the same 3-manifold have to be related to each other, by giving a handle interpretation of the Cerf theory, which describes how two Morse functions on the same manifold are related to each other (births, deaths, handle slides). Though this theorem does not have a useful corollary by itself, it gives a concrete way of looking at 3-manifolds as boundaries of the 4-ball with 2-handles. My own interest has been dealing with 4-manifold handlebodies to solve 4-manifold problems, where the interactions of 1- and 2-handles play particularly crucial role (e.g. my paper "On 2-dimensional homology classes of 4-manifolds" in "Math Proc. Camb. Phil. Soc." (1977))

Merely describing a handlebody of a manifold does not prove you a theorem, but it can be a first step towards solving a hard problem (usually in the positive); then you need to start applying operations like "handle slides" and some other techniques in creative ways to get a result. So what makes this a "calculus"? drawing handles? or sliding handles over each other? (if so, shouldn't we be calling this "Smale calculus"?) In retrospect, probably using the term "Kirby calculus" has been misleading, it ended up obscuring ideas of the proofs of some interesting 4-manifold theorems that used handlebodies, mostly by giving the nonspecialists an impression that a proof is merely an application of some picture stunts, or some kind of hocus-pocus magic moves. In reality those pictures are some useful tools used in low dimensional topology much like the inequalities in geometric analysis.

I wonder where the anonymous entries of Wikipedia gets the pop view of "Kirby Calculus", and even devoting a paragraph to a particularly simple handle operation with its own name "slam-dunk," and calling me as an expert on Kirby calculus. This is like calling a singer "an expert on yelling" instead of citing his songs, or describing a paper in differential equations "just calculus." If you call "chemistry" "alchemy" then be prepared to deal with peoples expectations that it is a practice of turning base metals into gold. So my tiring reply to the usual tiring question is "No I am not doing any kind of calculus, I am doing 4-manifolds, and yes I do use handles as basic tools".

¹More specifically, handles attached to the 4-ball.